

# Self-Shielding of X-rays and $\gamma$ -rays in Compact Sources

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## ABSTRACT

It is generally supposed that when the “compactness”  $l \equiv L\sigma_T/(rm_ec^3)$  in photons above the pair-production threshold is large, few  $\gamma$ -rays can escape. We demonstrate that even when  $l \gg 1$ , if the high energy and low energy photons are produced in geometrically-separated regions, many of the  $\gamma$ -rays can, in fact, escape. Pair-production along a thin surface separating the two sources creates enough Compton optical depth to deflect most of the low energy photons away from the high energy ones. Those few low-energy photons which penetrate the shielding surface are reduced in opacity by advection to large distance and small density, by relativistic beaming along the inner edge of the surface, and by Compton upscattering to higher energies inside the surface. The pairs in this surface flow outward relativistically, forming a structure resembling a pair-dominated mildly relativistic jet.

## 1. Introduction

One of the most basic concepts in the study of astrophysical  $\gamma$ -ray sources is that of “compactness” (see, *e.g.*, the review by Svensson 1986). Its importance stems from many causes, but one of its central implications has to do with the optical depth of a  $\gamma$ -ray source to pair production. Two photons with energies  $x_1$  and  $x_2$  (in units of  $m_ec^2$ ) may react to produce an  $e^\pm$  pair when their energies satisfy the relation  $x_1x_2 \geq 2/(1 - \cos\theta)$ , where  $\theta$  is the angle between their directions of motion. If a luminosity  $L$  in photons of energy just

above threshold is made isotropically within a source of size  $R$ , then the optical depth to pair production  $\tau_{\gamma\gamma}$  is greater than or of order unity when

$$l \equiv \frac{L\sigma_T}{Rm_e c^3} > 4\pi, \quad (1)$$

where  $\sigma_T$  is the Thomson cross section. The only caveat attaching to this statement is an order unity correction dependent on the details of the spectrum. In virtually all work since, it has been assumed that when  $l \gg 1$ , few high energy photons can escape the source region, although subsequent pair annihilation can partially restore them (*e.g.*, Guilbert, Fabian & Rees 1983; Svensson 1987; Blandford 1990). This argument has been used in many contexts because sources rich in  $\gamma$ -rays are very often copious sources of softer photons as well, and are also often variable enough to indicate causality bounds on the source size which result in large estimated compactnesses.

However, it is not always true that  $l \gg 1$  implies  $\tau_{\gamma\gamma} \gg 1$ . In particular, when the sources of high and low energy photons are geometrically separated, the very pair production which has been thought to prevent  $\gamma$ -ray escape can instead ensure it. Because high compactness generally also entails a significant Compton optical depth (Guilbert *et al.* 1983), an optically thick Compton scattering layer can form between the two source regions. High energy photons incident upon this surface from one side are either absorbed in pair production reactions, or else lose much of their energy by Compton recoil; low energy photons, which strike it from the other side, are scattered away, so that only a few cross the surface onto the side where the  $\gamma$ -ray source is located. The net result is that those high energy photons directed away from the surface can escape freely, while the energy of the  $\gamma$ -rays directed into the surface is transformed into a relativistic outflow of mixed  $e^\pm$  pairs and photons.

The remainder of this paper is devoted to working out a “toy-model” version of this idea so as to illustrate its qualitative features. It would be premature at this stage to use these ideas as the basis for a detailed model of any particular source or class of sources; our goal instead is to work out a number of gross features and scaling properties of this class of model to provide guidance for future, more realistic applications. Hence the following description is necessarily highly idealized.

## 2. Problem Definition

If the original photon sources radiate isotropically, there is no spherically symmetric system in which this screening mechanism may operate. The problem is that the high

energy photons must eventually pass through the low energy photon source region in order to escape. When they do so, they see an isotropic distribution of pair production partners, and the optical depth will be correctly approximated, at least to order of magnitude, by the usual scaling from the compactness.

However, there are a number of only slightly more complicated (and possibly even physically plausible) geometries which do permit operation of this mechanism. In this paper we will analyze one of them in a fashion that is deliberately as simple and idealized as possible. Real sources are undoubtedly more complicated. Our goal here is to highlight the basic physical principles underlying the scheme.

With accreting black hole systems (whether stellar binaries or active galactic nuclei) in mind, we choose a source geometry in which the high energy photons are generated isotropically with luminosity  $L_\gamma$  in a small spherical source of radius  $s$ , while the low energy photons (“X-rays”) are radiated isotropically with luminosity  $L_x$  from a ring of radius  $a$  whose center is the  $\gamma$ -ray source. In real sources, we expect that the regions of X-ray and  $\gamma$ -ray emission are likely to overlap. Nonetheless, internal gradients in the relative hardness of the spectrum are also quite likely. Our picture of complete separation between the  $\gamma$ -ray and X-ray sources is intended as an idealization which allows the basic idea to be seen more clearly through ease of calculation. In keeping with our effort to present the simplest possible case, we suppose that the spectra emitted by the small sphere and the ring have the same shape, but do not overlap in energy. That is, the sphere emits a spectrum (in erg/erg)  $\propto x^{-\alpha}$  for  $1 < x_l \leq x \leq x_{max}$ , while the spectrum of the ring is likewise  $\propto x^{-\alpha}$ , but for  $x_{min} < x \leq 1$ . Here and throughout the paper we give photon energies in units of  $m_e c^2$ .

This separation of high energy photons from low does not automatically guarantee that the  $\gamma$ -rays can escape from their source because the pair production cross section for pairs of photons well above threshold while small, is not zero. In order to avoid excessive pair production within the  $\gamma$ -ray source, the parameters must satisfy the condition

$$x_l > \left[ \left( \frac{1-\alpha}{1+\alpha} \right) \left( \frac{l_\gamma}{4\pi} \right) \frac{0.2}{x_{max}^{1-\alpha} - x_l^{1-\alpha}} \right]^{1/(2+\alpha)}. \quad (2)$$

Here, and elsewhere in the paper, when we need to integrate over the the  $\gamma - \gamma$  cross section, we approximate its energy dependence by

$$\sigma_{\gamma\gamma} \simeq 2g\sigma_T(x_1x_2)^{-1}, \quad (3)$$

where  $x_1$  and  $x_2$  are the energies of the two photons participating, and  $g \simeq 0.2$ . This is an approximation to the form given by Gould & Schreder (1967), corrected by Brown, Mikaelian & Gould (1973). When  $\alpha = 1$  (which is the center of the spectral distribution

for the AGN detected by EGRET: Hartman *et al.* 1994), the ratio  $(1 - \alpha)/(x_{max}^{1-\alpha} - x_l^{1-\alpha})$  becomes  $1/\ln(x_{max}/x_l)$ , so that  $x_l$  must be at least  $\sim (l_\gamma/4\pi)^{1/3}$ .

The question naturally arises whether such a hard spectrum could be created. Here we merely point out, in a very schematic way, a few possible mechanisms. In principle, a thermal source with  $kT \gg m_e c^2$  qualifies, though it may be difficult in practise to sustain such a source. Perhaps a more plausible mechanism would be for the  $\gamma$ -rays to be produced by inverse Compton scattering. If the seed photons have energy  $x_o < x_l^{-1}$ , and the electron distribution function is a power-law between  $\gamma_{min} = (x_l/x_o)^{1/2}$  and  $\gamma_{max} = x_o^{-1}$ , the resulting spectrum runs from  $\sim x_l$  to  $x_{max} = \gamma_{max} = x_o^{-1}$ , and there are *no* photons in the seed population that may pair produce with any of the  $\gamma$ -rays. Interestingly, electron distribution functions with  $\gamma_{min} \sim 100$  have also been suggested to exist in relativistic AGN jets in order to avoid pair annihilation constraints (Ghisellini *et al.* 1992) or to avoid Faraday depolarization (Wardle 1977).

In fact, the lower bound on the energies of photons produced in the  $\gamma$ -ray source can be softened somewhat when one takes into account a nonlinear transfer effect. Ordinarily  $\gamma$ -ray transfer is considered in the linear approximation; that is, it is assumed that the pair production opacity is fixed. However, when the number density of X-rays is not much greater than the number density of  $\gamma$ -rays, that assumption is not justified, and the transfer problem becomes nonlinear because the result of absorbing  $\gamma$ -rays in pair production events is to decrease the X-ray density as well, thereby diminishing the pair production opacity. To estimate the magnitude of this effect, we make the approximation that the greatest impact on the density of X-rays of energy  $x$  is due to pair production with  $\gamma$ -rays of energy  $\epsilon = 1/x$ , *i.e.* those right at threshold. If that is the case, the nonlinear response of the opacity permits  $\gamma$ -rays to escape—despite apparently large compactness—when  $n_x < \epsilon^2 n_\epsilon$ , where  $n_x$  is the number density per unit energy of the X-rays at dimensionless energy  $x$  and  $n_\epsilon$  is the number density per unit energy of the  $\gamma$ -rays. This condition is equivalent to requiring a spectral index  $\alpha < 0$  between  $x$  and  $\epsilon$ .

In at least two sources (CTA 102 and 3C 454.3: Blom *et al.* 1995), the observed spectra actually come close to meeting this criterion. In both, the observed spectral index is  $< 0.3$  from  $\sim 1$  keV to  $\sim 10$  MeV. Because there are no observations of these objects between 2 keV and 1 MeV, and the measurements below 20 MeV are only upper bounds, the spectrum could be even harder than this in the low-energy  $\gamma$ -ray band. In addition, since observed spectra integrate over the entire source, local regions within these sources could produce still harder spectra. Thus, sources such as these may be examples of places where there are indeed fewer low-energy pair-production partners than  $\gamma$ -rays.

### 3. Formation and Location of the Screening Surface

Suppose that both the X-ray source and the  $\gamma$ -ray source turn on gradually. The two classes of photons are initially well-mixed everywhere. As the photon densities increase, the pair production rate also increases.

When first created, the pairs move with the net momentum of the photons which produced them (the sum is dominated by the high energy photon, of course). From that point on, their motion is subject to Compton scatters against the photons and collisions with other pairs. Because these particles are relativistic, gravity is negligible unless these events take place close to the event horizon around a black hole. The X-rays see a significant Compton opacity (as a result of the pair production), while the  $\gamma$ -rays see both Compton opacity (in the Klein-Nishina regime) and pair-production opacity (which is greatest for the highest energy photons). Consequently, the local radiation force depends significantly on the optical depth from the point of observation to the various photon sources. Pairs close to the  $\gamma$ -ray source will tend to be pushed outward, while pairs just inside the X-ray ring will be pushed inward.

The pair densities in the zones closest to the two sources (just outside the origin, and just inside the ring) are further reduced by an additional optical depth effect: because the  $\gamma$ -rays and X-rays are no longer mingled, the pair production rate falls. Suppose we define these partially-evacuated zones by the surfaces on which the total optical depth from the nearest photon source is unity. Then the radiation forces identified in the previous paragraph lead to a continuous expansion of their volume.

Eventually, the bounding surfaces of the evacuated zones must meet. These merged surfaces are the screening surface. The electrons within this screening surface absorb momentum from the radiation by two mechanisms: by Compton scattering photons incident from either side, and by pair production inside the surface wherever the high and low energy photons can mix. If the electrons flow relativistically, the sign (relative to the direction of the electron velocity) of the net momentum exchange due to Compton scattering external photons can change from positive to negative. Momentum is also carried out of the surface by photons created within it which escape. The surface ceases to move when the net momentum transferred to it is directed entirely within its tangent plane, *i.e.* when the normal components of the momentum absorbed on either side exactly balance.

To find the location of this equilibrium surface, consider the limit in which the normal component of the incident radiation force is proportional to the normal component of the radiation flux. This limit is achieved when the albedo of the surface is small and it is optically thick.

Call the axis of the ring the  $z$ -axis. The screening surface is then defined by  $z_s(r)$ , where  $r$  is the cylindrical radius. By azimuthal symmetry, we need not consider azimuthal components of the radiation force. In the limit that the screening surface is very thin compared to the ring radius  $a$ , the condition that the net normal component be zero may be expressed as

$$\frac{L_\gamma}{L_x} \frac{|\hat{R} \cdot \hat{n}|^2}{r^2 + z_s^2} = \int_{-\Delta\phi}^{+\Delta\phi} \frac{d\phi}{2\pi} \frac{|\hat{R}_x(\phi) \cdot \hat{n}|^2}{d^2(\phi)}, \quad (4)$$

where  $\hat{R}$  is the unit vector in the spherical radial direction,  $\hat{n} = [-(dz_s/dr)\hat{r} + \hat{z}]/[1 + (dz_s/dr)^2]^{1/2}$  is the unit vector normal to the surface,  $\hat{R}_x(\phi) = [(r - a \cos \phi)\hat{r} - a \sin \phi \hat{y} + z_s \hat{z}]/d(\phi)$ , and  $d(\phi) = [r^2 - 2ar \cos \phi + a^2 + z_s^2]^{1/2}$ . The factors  $|\hat{R} \cdot \hat{n}|^2$  and  $|\hat{R}_x \cdot \hat{n}|^2$  appear because the pressure is proportional to the rate at which photons cross the surface times the component of their momentum parallel to the surface normal. The half-opening angle of the portion of the ring from which X-rays may arrive at  $(r, z_s)$  without being blocked by Compton opacity is  $\Delta\phi$ . The actual value of  $\Delta\phi$  as a function of  $z$  is hard to determine *a priori* because it depends on the diameter of the screening surface at lower altitudes. We adopt the simplistic approximation that the blocking is what would occur if the screening surface closer to the equatorial plane were simply a cylinder of radius  $a$  for  $r > a$ , or a cylinder of radius  $r$  for  $r < a$ . That is,  $\Delta\phi = \cos^{-1}(a/r)$  for  $r > a$ , or  $\Delta\phi = \cos^{-1}(r/a)$  for  $r < a$ .

Equation 4 implicitly defines an ordinary differential equation for  $z_s(r)$  parameterized by  $L_\gamma/L_x$ . The appropriate initial condition for this differential equation is  $r(z_s = 0)$  such that the radial component of the net force is identically zero. Solving equation 4 yields “hour-glass” shapes which widen into cones at large distance. The opening angles of these cones increase with increasing  $L_\gamma/L_x$  (Fig. 1).

In reality, the limit of large optical depth and zero albedo (or at least equal albedos for  $\gamma$ -rays and X-rays) is achieved at best imperfectly, but the degree of departure from this limit may be used to calculate an “effective”  $L_\gamma/L_x$ . In §5 we show that essentially all the  $\gamma$ -rays incident on the screening surface are absorbed; after calculating the bulk Lorentz factor of the flow (§6.1) and the characteristic temperature associated with the majority population of pairs within the surface (§6.2), one may compute the effective albedo to X-rays, and therefore the enhancement to the absorbed X-ray momentum flux.

#### 4. How the $\gamma$ -rays Escape: The Reduction of $\tau_{\gamma\gamma}$ on the Axis

If the photons all simply streamed freely away from their sources, the density of X-rays originating from angle  $\phi$  on the ring would fall  $\propto d^{-2}(\phi)$ . The optical depth to pair production for a  $\gamma$ -ray of energy  $x$  traversing the region would then be

$$\tau_{\gamma\gamma}^{(o)}(x) \simeq g \frac{1 - \alpha}{1 + \alpha} \frac{(x/2)^\alpha}{1 - x_{min}^{1-\alpha}} \frac{l_x}{4\pi}. \quad (5)$$

The expression we show assumes that the photon direction is parallel to the  $z$ -axis, but there is only a weak dependence on polar angle in the sense that the optical depth is smallest on the axis and greatest in the equatorial plane. Most of the optical depth is accumulated inside  $R \sim a$ ; at greater distances, the optical depth from  $R$  to  $\infty$  scales as  $(R/a)^{-2\alpha-3}$ . This scaling results from the effective increase in the pair production threshold as the photon trajectories become more and more parallel. Thus, if no other process happens first, virtually all emitted  $\gamma$ -rays would be transformed into pairs when  $l_x \gg 1$ .

However, the presence of intervening, relativistically streaming, electrons and positrons alters the density, energy spectrum, and direction of the X-rays. Call the bulk speed parallel to the surface  $\beta c$ , and the bulk Lorentz factor  $\Gamma$  (see §6.1 for estimates of their likely magnitude). We will label all quantities measured in the moving frame by primes.

When the Thomson depth  $\tau'_s$  of the screen (measured along the normal in the moving frame; this is not a Lorentz invariant because the direction of the apparent normal is not invariant) is more than a few, the photon number flux penetrating through the screen is reduced substantially. Scattering alone reduces their flux by  $\sim 1/\tau'_s$ . In addition, in the moving frame of the pair fluid, photons take on average a time  $\tau'_s h/c$  to diffuse through the screening surface. Here  $h$  is the transverse thickness of the screen. Photons which enter the screen at lab frame coordinate  $\eta$  (distance along the screen measured outward from the equatorial plane) then emerge at  $\eta + \beta\Gamma\tau'_s h$ . By the time they leave the surface, their density has been reduced by a further factor because the circumference of the screening surface has increased. If the surface is nearly conical (as Fig. 1 shows that it often is), the photon density is reduced by the factor  $1/[1 + \beta\Gamma\tau'_s h(d \ln r/d\eta)]$ . Moreover, their direction of motion has been changed so that (in the lab frame) their directions are concentrated within an angle  $\Gamma^{-1}$  of the local surface tangent. Some reflected photons may strike the surface multiple times (see Fig. 1); the importance of this effect increases with  $\Gamma$ . Note also that when  $\beta\Gamma\tau'_s h > \eta$ , the optical depth of a fluid element changes significantly from the beginning of this process to the end, so a suitably averaged value of  $\tau'_s$  should be used in a more careful calculation of this effect.

Compton scattering inside the surface causes the X-rays to lose some energy by recoil, but on balance to gain energy. Most of the electrons in the surface thermalize before annihilating (Lightman and Zdziarski 1987); their temperature in the bulk frame

$T'$  (measured in units of  $m_e c^2$ ) should be more than  $T_C$ , the Compton temperature of the incident X-rays, because they are also exposed to the much harder photons generated by the pair cascade inside the surface (§6.2). The (non-relativistic) Compton- $y$  parameter  $\tau_s'^2 T'$  will then very likely be large enough to substantially alter the X-ray spectrum inside the screening surface (see §5).

The net result of all these effects—the reduction in flux by scattering, the advection to larger distance, the redirection more nearly parallel to the surface, and the increase in mean photon energy—is to drastically reduce  $\tau_{\gamma\gamma}$  on the axis. Because most of  $\tau_{\gamma\gamma}^{(o)}$  is due to X-rays found within  $R \sim a$  of the origin, if the screening surface is Compton thick to at least  $z \sim a$ , it can effectively shield the  $\gamma$ -rays from the greatest part of the pair production opacity that would otherwise exist.

## 5. Pair Balance and the Compton Depth of the Critical Surface

Our next task, therefore, is to estimate the Compton depth of the screening surface as a function of position. Where the surface is optically thick to Compton scattering, it effectively shields those  $\gamma$ -rays whose paths are close enough to the axis that they miss the surface; in addition, those X-rays striking the surface where it is Compton thick are reflected outward, contributing to the X-ray flux seen in directions nearly tangent to the surface.

### 5.1. Pair Yield

To make this estimate, we must find the rate at which the energy of  $\gamma$ -rays striking the surface is transformed into pairs. The problem treated here is similar in many respects to the pair balance calculations of Svensson (1987) and Zdziarski (1988). In both our case and theirs, a pair cascade is initiated by high energy photons reacting with softer photons, and the subsequent generations of pairs lose energy primarily by inverse Compton scattering. Likewise, in all cases the pairs are trapped in the region (by magnetic fields? by scattering against plasma waves?) while photons can escape.

However, there are also a number of points of difference. Some may be accounted for by appropriate adjustments. In both Svensson’s and Zdziarski’s calculations, all the energy

injected into the region eventually found its way into the pair cascade; in our case, it is possible for some of the photons to cross the region without being absorbed (although in the cases of greatest interest, we expect the  $\gamma - \gamma$  optical depth to be large over the interesting range of photon energies). Their calculations assumed isotropic photons, while in our problem they are directed; this difference can be removed by appropriate renormalization of energies and compactnesses.

There are also a number of more significant contrasts, however. In their problems, the length scales governing the crossing time and the optical depth were the same as the length scale in the compactness; here they are different in the sense that both the crossing time and the optical depth are smaller in our context by the ratio  $h/a$ . In their case, the soft photons injected into the region were Comptonized by the full optical depth; in this case, they are Comptonized by reflection, so that most photons experience fewer scatterings than would be the case when they are emitted throughout the volume under consideration. The final significant contrast is that in Svensson’s calculation all Compton scattering takes place in the Thomson limit, while in Zdziarski’s calculation the most important Compton scattering events are in the Klein-Nishina regime; in our case, either limit may be appropriate, depending on the spectra of the two sources.

On the basis of these differences, we expect that the pair yield  $Y$ , the ratio between the total particle production rate and the absorbed  $\gamma$ -ray energy in units of  $m_e c^2$ , is likely to be somewhat smaller in our case than in theirs. This means that when  $l_\gamma \gg 1$ ,  $Y$  is at most  $\sim 0.1 L_T / L_\gamma$ , where  $L_T$  is that portion of the absorbed  $\gamma$ -ray luminosity which is deposited in pairs which scatter predominantly in the Thomson limit.

## 5.2. Fraction of $\gamma$ -rays absorbed

To find the portion of  $L_\gamma$  striking the surface which is absorbed, we must re-estimate the  $\gamma - \gamma$  optical depth along those directions including the effects of the screening surface. Because the energies and directions of the X-rays reflecting off the surface are quite different from what they were initially, the  $\gamma - \gamma$  optical depth cannot be directly computed on the basis of the X-ray intensity distribution radiated by the ring. We therefore divide the problem into two parts: the  $\gamma - \gamma$  optical depth due to never-scattered X-rays, and the  $\gamma - \gamma$  optical depth due to scattered X-rays.

Within the surface itself, never-scattered X-rays can be found in significant numbers only in an outer layer roughly one Compton depth thick. Therefore, most of the pair

production optical depth due to never-scattered photons is found on the portion of the ray beyond the surface. For rays encountering the surface at  $R \sim a$ , the contribution these photons make to  $\tau_{\gamma\gamma}$  is smaller than  $\tau_{\gamma\gamma}^{(o)}$  by a factor of a few; the reduction factor for rays striking the surface at larger distance is  $\sim (a/R)^{2\alpha+3}$  for the same reason that the outer contributions to  $\tau_{\gamma\gamma}^{(o)}$  scale this way.

Those X-rays reflected by Compton scattering are changed both in energy and direction. If  $T' < 1$ , the results of Lightman & Rybicki (1980) give at least an approximate description of the reflected spectrum in the moving frame. When the Compton  $y$ -parameter is larger than  $\simeq \ln(T'/x'_{min})$ , a fraction  $f_W = \max\{1, [(4T'/\ln(T'/x'_{min}))^{1/2}]\}$  of the photons are scattered into a Wien distribution at the temperature  $T'$ , with the remainder distributed into a power law with energy spectral index  $\simeq 0$ . Note that this fraction is independent of  $y$  (for values greater than the minimum) because we are interested in the reflected spectrum, not the spectrum of photons created inside the plasma. In the lab frame, the photon energies are increased by  $\Gamma(1 + \beta)$ , and, to the extent that  $\Gamma$  is greater than unity, they are beamed parallel to  $\hat{t}$ , the unit vector tangent to the surface.

Because the pairs in the screening surface move relatively slowly when  $R < a$  (see §6.1), relativistic boosting and beaming are not very important in that region. To estimate the pair production optical depth due to the scattered photons, we make the crude approximation that in this region the density of the photons is roughly what it would have been without reflection, and that their angular distribution is broad enough that few scattered X-rays move parallel to the  $\gamma$ -rays. The optical depth is then due to two contributions: that due to the flat power-law segment of the scattered photon spectrum, and that due to the Wien segment. Integrating these spectral shapes over the energy-dependent pair production cross section gives

$$\frac{\tau_{\gamma\gamma}^{(scatt)}}{\tau_{\gamma\gamma}^{(o)}} \simeq \frac{1 - f_W}{x_{min}(x/2)^\alpha \ln(4T'/x_{min})} + \frac{3f_W}{x_{min}x^{1+\alpha}T'}, \quad (6)$$

where the expression for the Wien contribution is valid for  $x > 2/T'$ . Not surprisingly, the result of Compton upscattering is to increase the opacity for lower energy  $\gamma$ -rays and decrease the opacity for higher energy  $\gamma$ -rays.

Farther out along the surface, relativistic effects can be expected to be more significant (though  $\Gamma$  is unlikely to ever be  $\gg 1$ , as demonstrated in §6.1). Because the direction of the surface tangent tends to a constant at large distances (Fig. 1), relativistic beaming will have the effect of reducing the angle between the  $\gamma$ -rays and X-rays, thereby diminishing the opacity.

Two other effects may also contribute to  $\gamma - \gamma$  opacity, but their magnitude is harder to estimate. Bremsstrahlung by the pairs may increase the number of low energy photons,

particularly in the side of the screen nearer the  $\gamma$ -ray source. While the total number of photons entering the screen is always likely to be dominated by the external X-ray source (unless  $L_\gamma/L_x \gg 1$ ), most of the externally created photons that penetrate far into the screen are upscattered to high enough energies that their pair production opacity is reduced. Synchrotron radiation may also contribute significant numbers of soft photons. Because high energy pairs are continually created by the absorption of high energy  $\gamma$ -rays, the steady-state electron distribution function will contain a nonthermal tail at high energies. However, the amplitude of this tail is very difficult to determine because of the many possible cooling mechanisms for these electrons (*e.g.* inverse Compton scattering, synchrotron radiation, Coulomb collisions with other electrons and positrons, plasma wave scattering).

Summarizing these arguments, we expect that if  $l_x/4\pi > 1$ , the fraction of the  $\gamma$ -ray energy striking the surface which is absorbed remains close to unity out to distances a few times  $a$ .

### 5.3. Equilibrium Compton optical depth

We may now estimate the optical depth as a function of position by balancing annihilation against pair production (*cf.* Guilbert *et al.* 1983). In the frame of the streaming pairs, the rate of pair production is given by the Lorentz-transformed rate at which  $\gamma$ -ray flux is absorbed times the pair yield  $Y$ . Note that the relevant flux comprises only those photons above the pair production threshold. In terms of the Compton depth of the surface, the positron density in the moving frame may be written as  $n'_+ = \tau'_s/(2\sigma_T h)$ . Thus, we find

$$\tau'_s = \left\{ \frac{Y}{\Phi(T')} \frac{1-\alpha}{x_{max}^{1-\alpha} - x_l^{1-\alpha}} \frac{f_\gamma l_\gamma}{4\pi} \left( \frac{ha}{R^2} \right)^2 [\Gamma(1 - \vec{\beta} \cdot \hat{R})]^3 |\hat{R} \cdot \hat{n}| \int_{\max(x_l, 1/[\Gamma(1-\vec{\beta} \cdot \hat{R})])} dx x^{-\alpha} \right\}^{1/2}, \quad (7)$$

where  $\Phi(T) \simeq 3/8$  is the pair annihilation rate at temperature  $T$  in units of  $\sigma_T c$ , and  $f_\gamma$  is the fraction of the  $\gamma$ -ray energy absorbed. In the equatorial plane, where  $\Gamma \simeq 1$ , we expect  $\tau'_s \simeq [Y f_\gamma l_\gamma / (4\pi)]^{1/2} (h/R)$ . Equation 7 shows, however, that the optical depth should typically fall rapidly for  $\eta > a$ : as the pairs stream out, their density falls; at large distances the photon directions become nearly parallel, so that the pair creation threshold rises and the primary production rate falls; and the photon directions become nearly tangent to the screening surface, so that the rate at which photons enter the surface falls. In other

words, when  $l_x/4\pi > 1$  and  $l_\gamma/4\pi \gg 1$ , the screening surface is optically thick to Compton scattering out to distances at least  $\sim a$ . It is in these circumstances that the surface is able to effectively protect the  $\gamma$ -rays moving within the open cone from pair production with the X-rays.

## 6. Bulk Properties of the Surface

### 6.1. Flow Speed

The momentum of the flow along the surface is carried both by the pairs and by the photons trapped within the surface. New momentum is added by the arrival of both  $\gamma$ -rays and X-rays; to the extent that X-rays are reflected, particularly with additional energy, they carry away momentum. Photons produced by annihilation or bremsstrahlung can remove momentum as they escape from the flow. At the same time, the inertial density of the surface is increased by energy absorbed from the incident X-rays and  $\gamma$ -rays, and diminished by escaping photons. In the steady state and assuming pure absorption (and no intrinsic radiation), the conservation equations for momentum along the surface and the inertial density combine to form the pair

$$\frac{4\pi c^3 \beta \rho h \Gamma}{[1 + (dz_s/dr)^2]^{1/2}} \frac{\partial \beta}{\partial r} = \frac{f_\gamma L_\gamma}{r^2 + z_s^2} |\hat{n} \cdot \hat{R}| \left[ |\hat{t} \cdot \hat{R}| - \beta \right] + \int_{-\Delta\phi}^{+\Delta\phi} \frac{d\phi}{2\pi} L_x \frac{|\hat{n} \cdot \hat{R}_x(\phi)|}{d^2(\phi)} \left[ |\hat{t} \cdot \hat{R}_x(\phi)| - \beta \right] \quad (8)$$

and

$$\frac{4\pi c^3 \beta}{[1 + (dz_s/dr)^2]^{1/2}} \frac{\partial(\rho h \Gamma)}{\partial r} = \frac{f_\gamma L_\gamma}{r^2 + z_s^2} |\hat{n} \cdot \hat{R}| + \int_{-\Delta\phi}^{+\Delta\phi} \frac{d\phi}{2\pi} L_x \frac{|\hat{n} \cdot \hat{R}_x(\phi)|}{d^2(\phi)}. \quad (9)$$

Derivatives with respect to distance along the surface are related to derivatives with respect to  $r$  by the factor  $[1 + (dz_s/dr)^2]^{1/2}$ . The factors proportional to  $|\hat{n} \cdot \hat{R}|$  and  $|\hat{n} \cdot \hat{R}_x|$  are the geometrical corrections for  $\gamma$ -ray and X-ray flux, respectively, entering the surface. The dot products of  $\hat{R}$  and  $\hat{R}_x$  with  $\hat{t}$  are the geometrical corrections accounting for the portion of the incident momentum (from  $\gamma$ -rays and X-rays, respectively) parallel to the surface. The subtracted factor of  $\beta$  is required because the energy absorbed along with momentum increases the inertia density, so that the speed does not increase unless the geometric factor exceeds  $\beta$ .

Solutions of these equations are specified by two initial conditions, one on  $\beta$  and the other on  $\rho h \Gamma$ . At  $r(z_s = 0)$ , it is obvious by symmetry that  $\beta$  must be zero (for

numerical solutions we set it to 0.01 in order to avoid artificial divergences). In appropriate dimensionless units, the inertia density  $\rho h\Gamma = (\mu_e/m_e)\tau_T/l_x$ , where  $\mu_e$  is the inertia per electron.  $\mu_e$  can be greater than  $m_e$  due both to the relativistic motions of the electrons and the effective inertia of the radiation trapped in the surface. Again to avoid numerical problems, we set this initial condition to a small fraction of unity; in practise, it rises to its asymptotic value so swiftly that there is little dependence on exactly how we choose this initial condition provided it is small.

Numerical solution of equations 4, 8, and 9 demonstrates that solutions extending to infinity exist for all  $L_\gamma/L_x > 0.08$ . When this ratio is smaller, the surface closes over the origin, and  $\gamma$ -rays cannot escape freely in any direction. We expect that in this case the ultimate structure would be a quasi-cylindrical pair-rich wind, but we have not investigated it in detail.

For  $L_\gamma/L_x > 0.08$ , the surface becomes asymptotically conical with an opening angle that increases with increasing  $L_\gamma/L_x$  (Fig. 1). When  $L_\gamma/L_x > 10$ , the half opening angle is nearly  $\pi/2$ , *i.e.* the only region from which X-rays are not excluded is a thin wedge in the equatorial plane. However, because a careful calculation of the pair balance is beyond the scope of this paper, it is possible that further collimation toward the polar direction occurs in the region  $a < r < few \times a$ . When the screening surface becomes optically thin (at  $r \sim a$ ?), the complete absorption approximation is no longer appropriate. Instead, the radiation force is simply proportional to the flux times the opacity. Because Klein-Nishina effects reduce the opacity of the  $\gamma$ -rays, the effect of the X-rays is comparatively enhanced. Provided  $r$  is not too much larger than  $a$ , that translates to an increase in the component of the radiation force parallel to  $\hat{z}$ .

Both the tangential speed  $\beta$  and the inertia density  $(\mu_e/m_e)\tau_T/l_x$  quickly reach asymptotic values (Figs. 2 and 3). At first glance it might seem surprising that the asymptotic speed is only mildly relativistic unless  $L_\gamma \gg L_x$ . After all, when a high energy  $\gamma$ -ray combines with an X-ray to create a pair, the electron and positron both have very large Lorentz factors. However, in this model most of the pairs are created where  $z_s/a < 1$ , and the momenta of the  $\gamma$ -rays and X-rays are almost oppositely directed. Consequently, the plasma begins with a fairly large ratio of inertia density to momentum, and by the time the  $\gamma$ -rays are more nearly parallel to the flow, little can be done to accelerate it. It is possible that greater speed might be found with allowance for intrinsic radiation. However, the same radiation which keeps the inertia density small also removes momentum from the flow. We have modelled this effect by a phenomenological non-zero albedo, and confirmed that  $\beta$  depends only weakly on the size of the albedo (in fact, increasing the albedo to 0.5 actually *decreases* the asymptotic  $\beta$  slightly). On the other hand, if for some reason the

albedo decreased with distance, the asymptotic speed might be increased.

The asymptotic inertia density quickly reaches a constant value because the rate of energy absorption by the surface decreases rapidly with increasing distance. The flux striking the surface falls so quickly both because of increasing distance and because the surface becomes nearly tangential to the photon paths it intercepts.  $(\mu_e/m_e)\tau_T/l_x$  is a very slowly increasing function of  $L_\gamma/L_x$ ; for  $L_\gamma/L_x \sim 1$ , it is  $\simeq 0.3$ . Provided  $\mu_e/m_e$  is not too large, the surface is indeed Compton thick whenever  $l_x$  is significantly greater than unity. It should be borne in mind, however, that our model equations do not include the effect of photon leakage out of the surface as its optical depth decreases at large distance. Consequently, these asymptotic values of the inertia density are likely to be overestimates at  $(r^2 + z_s^2)^{1/2} \gg a$ . At larger distances, the flow speed could well be larger than our model predicts because the pairs are not forced to share their energy with so many trapped photons.

## 6.2. Pair Temperature

The temperature of those pairs which have cooled into a thermal distribution is determined primarily by Comptonization balance. Generically, we expect bremsstrahlung to play a secondary role in cooling: relative to inverse Compton scattering of the incoming X-ray photons, the bremsstrahlung luminosity when  $T' \sim 1$  is  $\sim \alpha_{fs}(\tau'_s/l'_x)(a/h)$ , where  $\alpha_{fs}$  is the fine structure constant and  $l'_x$  is the X-ray compactness on the scale  $a$  as viewed from the frame of the screening surface. The local photon spectrum participating in this Comptonization equilibrium combines incident  $\gamma$ -rays which have avoided pair production, incident X-rays which have penetrated into the surface with little energy change, photons originally in the X-ray power-law which have scattered sufficiently against the electrons to enter a Wien distribution, and the smaller number of photons which have undergone large increases in energy by Compton scattering against nonthermal electrons. Because the effectiveness of scattering declines for those photons which interact in the Klein-Nishina regime, the most likely temperature is a fraction of unity, that is,  $\sim 100$  keV.

## 6.3. Geometrical Thickness

When  $\tau'_s > 1$ , the annihilation time for an electron is shorter than the sound wave crossing time. The thickness of the surface is therefore controlled more by pair balance

(that is, radiation transfer) than by pressure balance. Suppose, then, that  $x'_*$  is the  $\gamma$ -ray energy in the bulk frame at which  $Y(x')L'_\gamma(x')$  is greatest. The pair balance as a function of distance  $\xi$  into the surface is then determined by two coupled differential equations which describe the radiation transfer:

$$\mu' \frac{dn'_\gamma}{d\tau'} = - \left( \frac{\zeta \Phi n'_{xo}}{2x'_* Y} \right)^{1/2} \left( \frac{\tau'}{\tau'_s} \right)^{1/2} n'^{1/2}_\gamma \quad (10)$$

and

$$\frac{d\tau'}{d\xi} = 2n'_+ \sigma_T \quad (11)$$

In these equations,  $\mu' = |\hat{n}' \cdot \hat{R}'|$ ,  $n'_\gamma$  is the number density in the bulk frame of photons of energy  $x'_*$ ,  $n'_+$  is the density of positrons, and  $\zeta$  is the pair production cross section at  $x'_*$  in units of  $\sigma_T$  averaged over the X-ray spectrum. The first equation describes the transfer of  $\gamma$ -rays subject to absorption by pair production with X-rays. The second equation defines the Compton optical depth scale relative to the distance scale. The Compton opacity is proportional to  $n'^{1/2}_\gamma$  because the pair density (assuming pair balance) is proportional to the square root of the  $\gamma$ -ray density:

$$n'_+ = \left( \frac{\zeta n'_\gamma n'_{xo} \tau' x'_* Y}{2\tau'_s \Phi} \right)^{1/2} \quad (12)$$

Here we have also made the approximation that Compton scattering opacity dominates pair production opacity for X-rays, so that the X-ray number density decreases linearly with Thomson depth  $\tau'$  from  $n'_{xo}$  at the edge where they strike the surface, to zero at the edge where the  $\gamma$ -rays enter. However, the mean energy of the X-rays changes as a result of Compton scattering; this variation is absorbed into  $\zeta$ .

Equation 10 is readily solved in terms of  $\tau'$ :

$$n'_\gamma = \left[ n'_{\gamma o} - \frac{\tau'^{3/2}}{3\tau_s'^{1/2} \mu'} \left( \frac{\zeta \Phi n'_{xo}}{2x'_* Y} \right)^{1/2} \right]^2, \quad (13)$$

where  $n'_{\gamma o}$  is the number density of  $\gamma$ -rays at the  $\tau = 0$  edge. Substituting this result in the opacity differential equation, two characteristic length scales are revealed:

$$\lambda_1 = \frac{1}{\sigma_T} \left( \frac{\Phi \tau'_s}{2\zeta x'_* Y n'_{xo} n'_{\gamma o}} \right)^{1/2} \quad (14)$$

and

$$\lambda_2 = \frac{6\mu' \tau'_s}{\zeta n'_{xo} \sigma_T}. \quad (15)$$

The first of these describes the characteristic thickness of the surface's edge on the side facing the  $\gamma$ -ray source; the second describes the thickness on the side facing the X-ray source. Typically we expect  $\lambda_1$  to be rather greater than  $\lambda_2$  because  $n_{xo} \gg n_{\gamma o}$ . The total thickness is, of course, of order their sum.

Because both scales depend on  $\tau'_s$ , which, in turn, depends on the sum of the two lengthscales, it is possible to use these relations to solve explicitly for  $h/a$ . This procedure is simplest at  $z_s = 0$  where  $\beta = 0$ , so the relativistic transformations are null. When, in addition,  $1 - r_s/a \ll 1$  (a condition which applies whenever  $L_\gamma/L_x$  is not too small), the incident X-ray density is

$$n_{xo} \simeq \frac{l_x}{a\sigma_T[1 - (r_s/a)^2]}. \quad (16)$$

Using this X-ray density, we find the optical depth at  $z_s = 0$ :

$$\tau_s = \frac{h}{a} \left( \frac{Y f_\gamma l_\gamma}{4\pi\Phi} \right)^{1/2}. \quad (17)$$

Now the sum  $(\lambda_1 + \lambda_2)/a$  can be computed, and solved for  $h/a$ :

$$\frac{h(z_s = 0)}{a} = \left( \frac{\pi\Phi f_\gamma}{Y l_\gamma} \right)^{1/2} \left[ \frac{1 - (r_s/a)^2}{\zeta l_x} \right] \left\{ 1 - \frac{6}{\zeta l_x} \left[ \left( \frac{a}{r_s} \right)^2 - 1 \right] \left( \frac{Y f_\gamma l_\gamma}{4\pi\Phi} \right)^{1/2} \right\}^{-2}. \quad (18)$$

Thus, we can expect the surface to be quite thin near  $z_s = 0$  throughout the interesting regime, *i.e.*, when both  $l_x$  and  $l_\gamma$  are greater than unity. However, as the surface accelerates away from the photon sources, relativistic effects reduce the photon densities in the moving frame, causing the surface to thicken somewhat. The condition that the final factor in equation 18 should be near unity is equivalent to  $\lambda_2 \ll \lambda_1$ .

## 7. Summary

We have shown that when  $L_\gamma/L_x > 0.08$ , the pairs that are created by  $\gamma$ -ray – X-ray reactions are squeezed into a screening surface shaped like an hour-glass at relatively small distances, but which becomes asymptotically conical at large distances. As equation 7 demonstrates, when  $l_\gamma \gg 1$  and  $l_x$  is large enough to make the surface optically thick to pair production, the screening surface acquires a significant Compton optical depth at least to heights  $z_s \sim a$ . When that occurs, most of the incident X-rays are scattered back outside the surface, although the relativistic bulk motion will beam them within an angle  $1/\Gamma$  of

the local surface tangent. As a result, very few X-rays penetrate into the central region, and the  $\gamma$ -rays directed within the surface are free to escape. This conclusion differs from that of Bednarek (1993), who found that a compact X-ray ring would lead to  $\gamma$ -ray absorption, because he neglected the Compton scattering of X-rays by the pairs.

On the other hand, most of the energy of the  $\gamma$ -rays initially directed into the screening surface is absorbed. Thus, this mechanism has the net effect of using a fraction of the  $\gamma$ -rays (the covering fraction of the surface around the origin) to preserve the remainder. When  $L_\gamma/L_x \sim 1$ , the two fractions are comparable.

Both the escaping  $\gamma$ -rays and the X-rays are collimated by this process. The  $\gamma$ -rays can only escape in those directions not covered by the screening surface. On the other hand, those X-rays initially directed toward the surface are focussed into directions nearly tangent to the surface, and also inverse Compton scattered to higher energies. Therefore, when our line of sight lies nearly parallel to the asymptotic direction of the screening surface, we should see an X-ray spectrum which is both quite strong and quite hard. Thus, objects in which this process operates can be expected to have strikingly different high energy spectra when viewed from different directions.

Because the asymptotic direction of the screening surface changes with  $L_\gamma/L_x$ , it is possible for fluctuations in  $L_\gamma/L_x$  to strongly modulate the observed X-rays. When  $L_\gamma/L_x$  is small, our line of sight is likely to lie outside the screening surface, so we see the unmodified X-ray spectrum; when  $L_\gamma/L_x$  is large, the screening surface swings outward, cutting off our view of the X-rays; for some range of intermediate values, our line of sight will lie close enough to the direction of the surface that we will see the beamed and upscattered X-rays. It is possible that these effects have been observed in the BL Lac object PKS 2155-304. Ordinarily its spectral index above a few keV is in the range 1 – 2 (Sembay *et al.* 1993), but in one observation (Urry & Mushotzky 1982) the spectral index from 15 keV up to the sensitivity limit of the instrument,  $\simeq 40$  keV, was  $\simeq -1.5$ ! Such behavior might be explained if an excursion to especially large  $L_\gamma/L_x$  opened the screening surface wide enough for our line of sight to be nearly aligned with it.

In many of the most powerful high energy  $\gamma$ -ray sources known there is independent evidence strongly suggesting that much of the radiation comes from relativistic jets (Hartman *et al.* 1994; Dermer, Schlickeiser & Mastichiadis 1992; Blandford 1993; Sikora, Begelman, & Rees 1994). Some have argued that the existence of strong high energy  $\gamma$ -radiation in these objects is itself evidence for relativistic motion (*e.g.* Zdziarski and Krolik 1993; Dondi & Ghisellini 1995). While the mechanism we described here certainly does not argue against relativistic motion in the source, it does create a loophole in the arguments for relativistic motion solely on the basis of high energy compactness.

Our point of view is that in more realistic pictures of these sources, *both* relativistic motion of the source and optical depth effects such as the ones we have discussed in this paper may operate. One might easily imagine, for example, that the toy geometry we have explored here should, in real sources, be modified to take into account a relative velocity between the  $\gamma$ -ray source and the X-ray source which is very likely relativistic, and may or may not be aligned with the symmetry axis. Such relative motion could beam the  $\gamma$ -rays away from the X-ray source, so that a smaller fraction of the  $\gamma$ -rays are used to produce a shielding wall which is still sufficiently optically thick to be effective. This effect could help solve an otherwise troubling problem in jet models of  $\gamma$ -ray production in AGN: that if  $\gamma$ -rays are produced too close to the center of the system, pair production on X-rays would lead to a saturated pair cascade and the production of a much larger X-ray luminosity (through inverse Compton scattering by the pairs that are produced in the cascade) than is observed (Ghisellini & Madau 1995).

These effects have another significant consequence: there is a collimated outflow in the screening surface whose luminosity is equal to the absorbed  $\gamma$ -ray luminosity, which is likely to be an interesting fraction of  $L_\gamma$ . This outflow, composed of a mixture of electron-positron pairs and photons of comparable energy, is automatically mildly relativistic, with a modest bulk  $\Gamma$ . Particularly when  $L_\gamma/L_x < 1$  so that the opening angle is comparatively small, these outflows have many of the characteristics of the relativistic jets thought to be responsible for much of the lower-frequency radiation in these objects.

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### Figure Captions

Figure 1      Projection of the screening surface into the  $r - z$  plane. From top to bottom, the curves are for  $L_\gamma/L_x = 0.1, 1.0, 10$  and  $100$ . All assume  $\Delta\phi$  is determined by a projection of the structure into the equatorial plane. The radius of the X-ray emitting ring  $a$  has been used as the unit of distance.

Figure 2       $\beta$  as a function of  $R = (r^2 + z_s^2)^{1/2}$ . The four curves are for  $L_\gamma/L_x = 0.1, 1, 10$ , and  $100$ ; the asymptotic value of  $\beta$  increases (slowly) with increasing  $L_\gamma/L_x$ .

Figure 3      The inertia density  $(\mu_e/m_e)\tau_T/l_x$  as a function of  $R = (r^2 + z_s^2)^{1/2}$ . The four curves are for  $L_\gamma/L_x = 0.1, 1, 10$ , and  $100$ ; the asymptotic value of  $\tau_T/l_x$  increases (very slowly) with increasing  $L_\gamma/L_x$ .





